Unified Mathematics of Geometry and Music

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ABSTRACT

Unified mathematical models must establish numerical connections between all types of physical phenomenon. Inherent relationships between geometry and music are shown through the inscription of regular polygons within a unit circle.

SUMMARY

Hertz is the standard measurement of frequency in sound, calculated by the number of times a sound's waveform or vibration pattern repeats in a single second. Although the modern tuning standard for the note A is 440hz, "true" Pythagorean tuning originally set the note A to 432hz. To see the direct connections between geometry and sound, Pythagoras' original chromatic tuning scale must be used.

Notes will repeat with a uniform ratio called "octaves," calculated by doubling or halving any hertz value. There are a total of 12 notes, so the 13th note in a scale will be a repeat or up-octave of the first. For example, A4 = 216hz, A5= 432hz, and A6 = 864hz, where each number following the letter denotes the octave range the frequency belongs to. The note C4 is commonly referred to as "middle C" due to the range best fitted for human hearing. Pythagoras first discovered this pattern by noticing that plucking a string sustained a vibration or note and that plucking a string half as long sustains the same note at a higher pitch (See Figure 1). He used the same division to identify the frequencies of all other notes on the scale. Today, these ratios have been further honed and "equally tempered" to a greater degree of accuracy, meaning the frequencies shown are more symbolic references to the notes themselves. Figures 2-10 demonstrate how these values can be plotted around a circle, connecting the musical scale to arc lengths, interior angles, sums of angles, and natural progressions of regular polygons.

RESULTS

Each note in the scale has 2 corresponding degree references within one unit circle (circumference = 360), as the unit circle encompasses 2 octaves. Whole and half steps are smaller distances used in music theory to describe the gaps between notes within the same octave. For example, the half step gap from A5 (432hz) to A#5 (450hz) would be 30° on the unit circle, symbolizing an angle as the most elementary geometrical unit. The 360° unit circle is fundamental to our understanding of geometry and sound as each progressive note will correspond to a respective polygon (Fig. 2-8).

"Gap notes" are empty spaces where there is no sharp or flat since the distance between the two whole notes (B-C and E-F) is only a half step, explained by the lack of intersection points when inscribing regular polygons within the unit circle. There are exceptions to this rule in deeper levels of music theory. The seminal note of the Flower of Life (F#: 720hz) lands perfectly on the circle at 186.32 degrees, referencing light speed (186.28 x 10^3 miles per second) +/- .00001.

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The following table will show the harmonic relationships between notes in the chromatic scale, their degree references, and their resulting polygonal geometries. For example, a B5 note following an A5 is 1.0833 x 432hz = 468hz, which lands at 300° (0.8333 of 360°) on a circle. The arc length is calculated by measuring the distance around the circle from the starting point (note A5 – B5 or 360°-300°). Therefore, the B5 (468hz) creates an equilateral triangle (see Figure 4). All further progressions of regular polygonal geometry follow with perfect symmetry (See Figure 2). For example, the note C produces an inscribed square since the arc length of 90° is 1/4th of the 360° unit circle. The note C# produces an inscribed hexagon since the arc length of 60° is 1/6th of the unit circle, meaning there would be a total of 6 equal sides to the polygon. The pattern continues through the audible spectrum of sound and is calculated for one full octave scale in the table below. Table 2 collects scale positions, degree references, decimal references, and doubling hertz frequency values for all 12 notes.

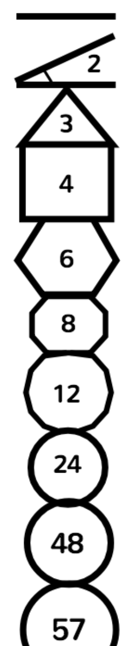
Arc Length	Degrees	Decimal (°/360)	NOTE	Length (unit)	Sum of Angles	HERTZ	Decimal (hz/432)	GEOMETRY
0°	360°	1.00	A_5	12	0°	432hz	1.00	
30°	330°	.9166	$A\#_5$	11/12	30°	450hz	1.0416	2
60°	300°	.8333	B_5	10/12	180°	468hz	1.0833	3
90°	270°	.7500	C_5 GAP I	9/12	360° GAP: 540°	504hz	1.1666	4
60°	240°	.6666	$C\#_5$	8/12	720°	540hz	1.2500	6
45°	225°	.5833	D_5	7/12	1080°	576hz	1.3333	8
30°	210°	.5620	$D\#_5$	6.75/12	1800°	612hz	1.4166	12
15°	195°	.5406	E_5	6.48/12	3960°	648hz	1.5000	24
7.5°	187.5°	.5208	F_5	6.24/12	8280°	684 hz	1.5833	48
6.32°	186.32°	.5175	$F\#_5$	6.18/12	9900°	720hz	1.6666	57
4.76°	184.76°	.5104	G_5	6.124/12	13140°	756hz	1.7500	75-sides
2.4°	182.4°	.5052	$G\#_5$	6.0624/12	26640°	792hz	1.8333	150-sides
1.2°	181.2°	.5033	GAP II	6.0399/12	53640°	828 hz	1.9166	300-sides
NEXT OCTAVE	180°	.50	A_6	6/12	0	864hz	2.00	1

Table 1

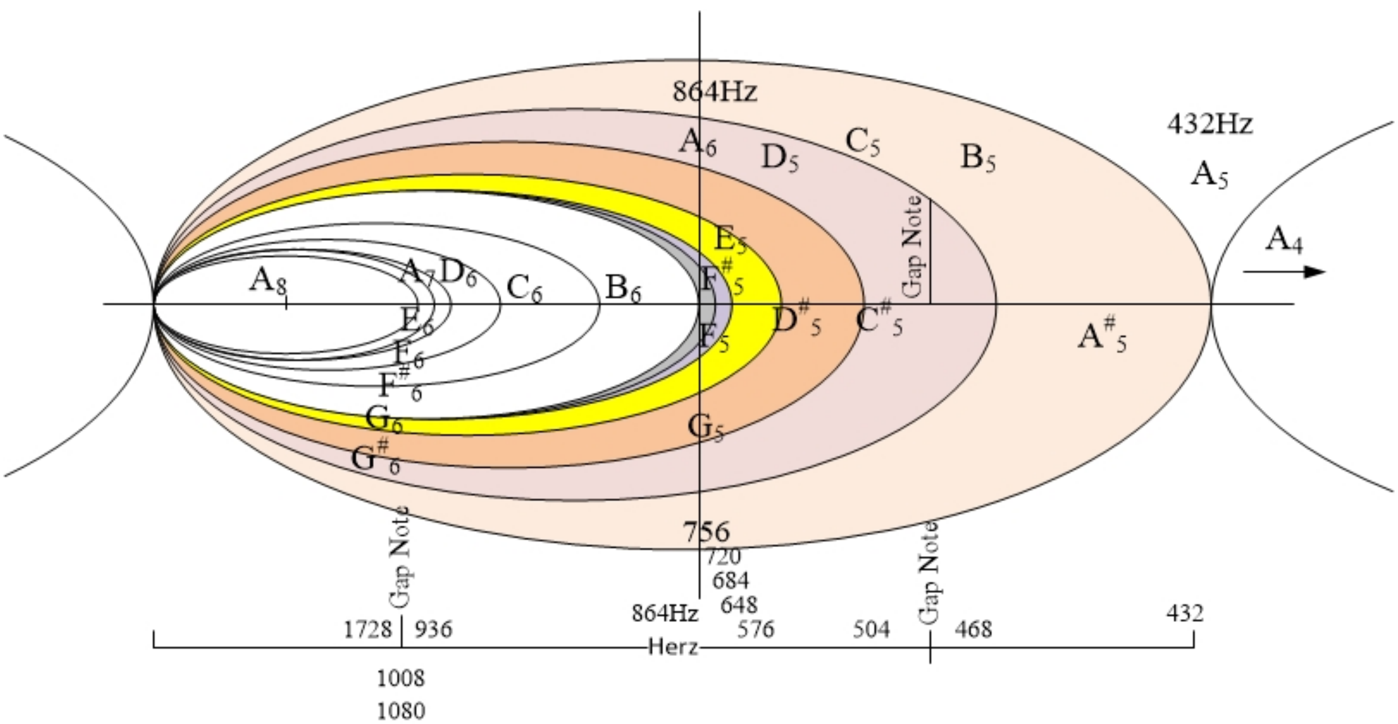
THE CHROMATIC SCALE (12 NOTES/OCTAVE)

Polygon (Geometry)	PENTA	Square	PENTA	Hexagon	8-gon	12-gon	24-gon	48-gon	57-gon	75-gon	150-gon	300-gon	CIRCLE	LINE	ANGLE	Triangle
Position in A Scale		1		2	3	4	5	6	7	8	9		10		11	12
Position in C Scale		4		5	6	7	8	9	10	11	12		1		2	3
Arc Length	72	90	72	60	45	30	15	7.5	6.3157	4.8	2.4	1.2	0	18	30	60
Arc Position	288	270	252	240	225	210	195	187.5	186.32	184.8	182.4	181.2	180	162	150	120
Decimal Reference (360)	0.8000	0.7500	0.7000	0.6667	0.6250	0.5833	0.5417	0.5208	0.5175	0.5133	0.5067	0.5033	0.5000	0.4500	0.4167	0.3333
Decimal Reference (432)	1.1250	1.1666	1.2083	1.2500	1.3330	1.4166	1.5000	1.5833	1.6660	1.7500	1.8333	1.9166	2.0000	1.0208	1.0417	1.0833
Hertz Frequency	486	504	522	540	576	612	648	684	720	756	792	828	864	882	900	936

NOTE	GAP C	C	GAP C	C#	D	D#	E	F	F#	G	G#	GAP A	A	GAP A	A#	B
Octave -1					9											
Octave 0					18								27			
Octave 1					36											
Octave 2									45				54			
Octave 3		63			72				90		99		108			117
Octave 4		126		135	144	153	162	171	180	189	198	207	216		225	234
Octave 5	243	252	261	270	288	306	324	342	360	378	396	414	432	441	450	468
Octave 6	486	504	522	540	576	612	648	684	720	756	792	828	864	882	900	936
Octave 7	972	1008	1044	1080	1152	1224	1296	1368	1440	1512	1584	1656	1728	1764	1800	1872
Octave 8	1944	2016	2088	2160	2304	2448	2592	2736	2880	3024	3168	3312	3456	3528	3600	3744
Octave 9	3888	4032	4176	4320	4608	4896	5184	5472	5760	6048	6336	6624	6912	7056	7200	7488
Octave 10	7776	8064	8352	8640	9216	9792	10368	10944	11520	12096	12672	13248	13824	14112	14400	14976
Octave 11	15552	16128	16704	17280	18432	19584	20736	21888	23040	24192	25344	26496	27648	28224	28800	29952
Octave 12	31104	32256	33408	34560	36864	39168	41472	43776	46080	48384	50688	52992	55296	56448	57600	59904

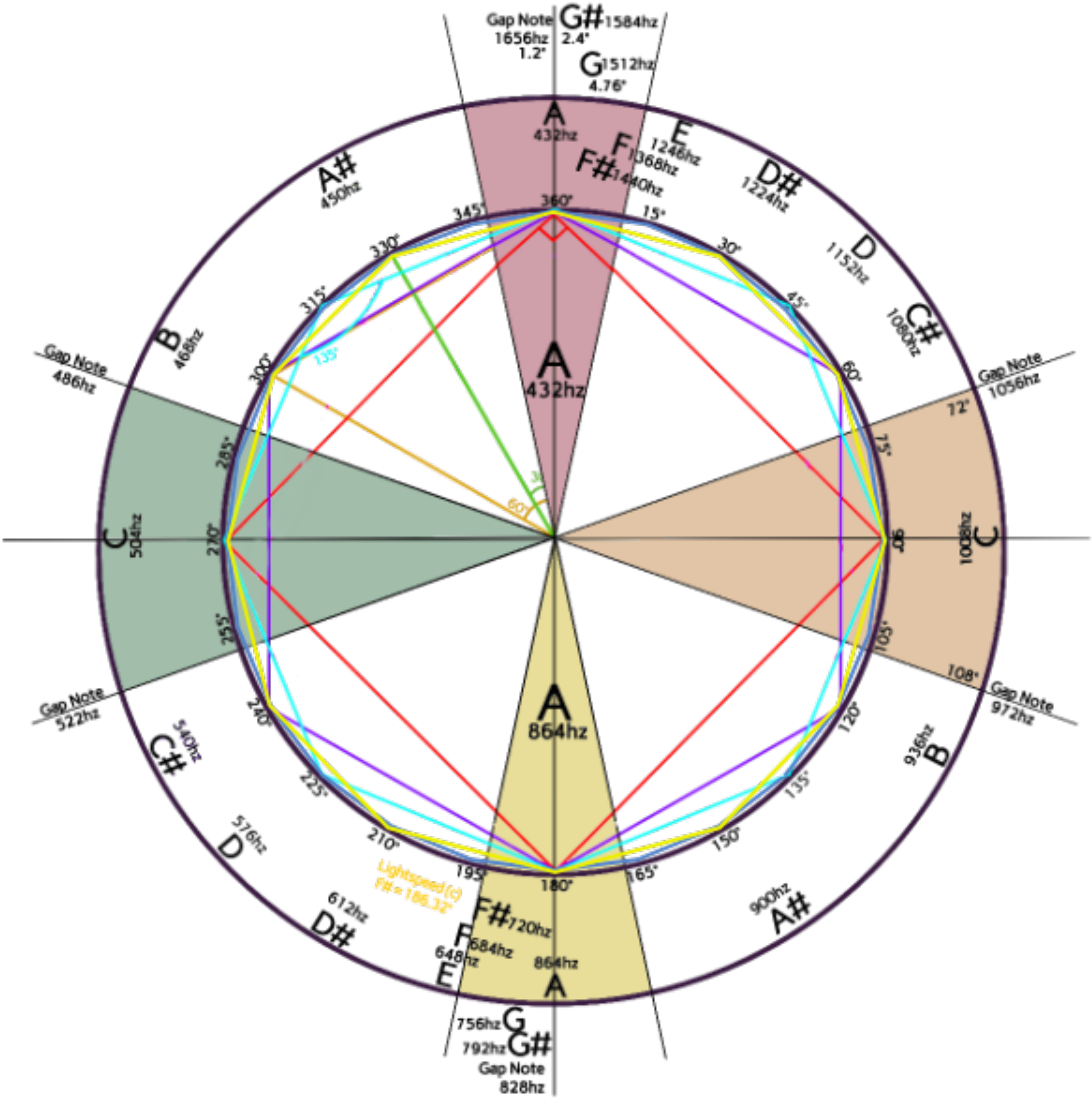
PERFECT THIRD		E		F	F#	G	G#	A	A#	B	C		C#		D	D#
PERFECT FIFTH		G		G#	A	A#	B	C	C#	D	D#		E		F	F#

Table 2



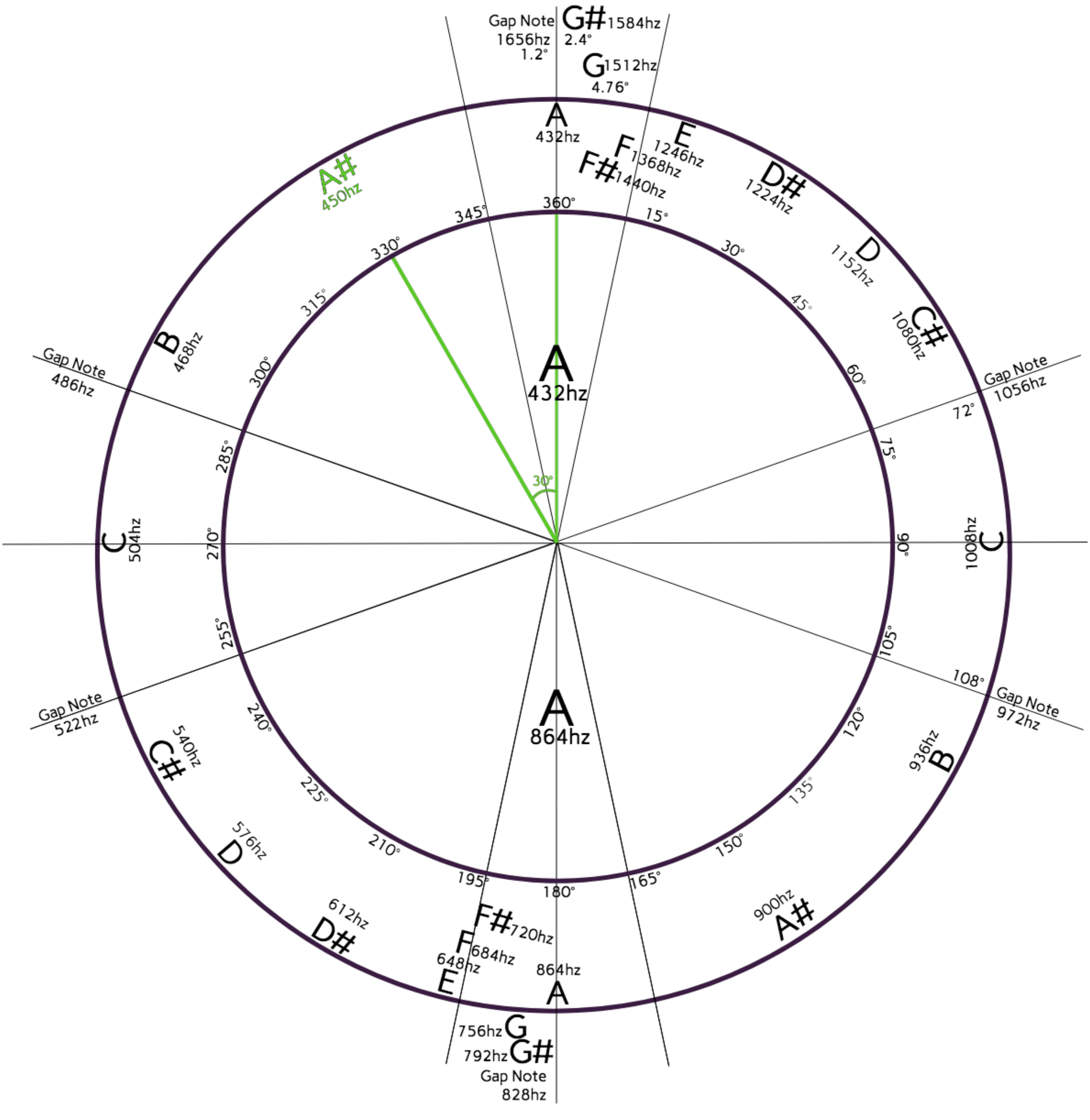
Two whole octaves depicted; moving up the note scale requires progressively smaller distances like guitar frets

Figure 1



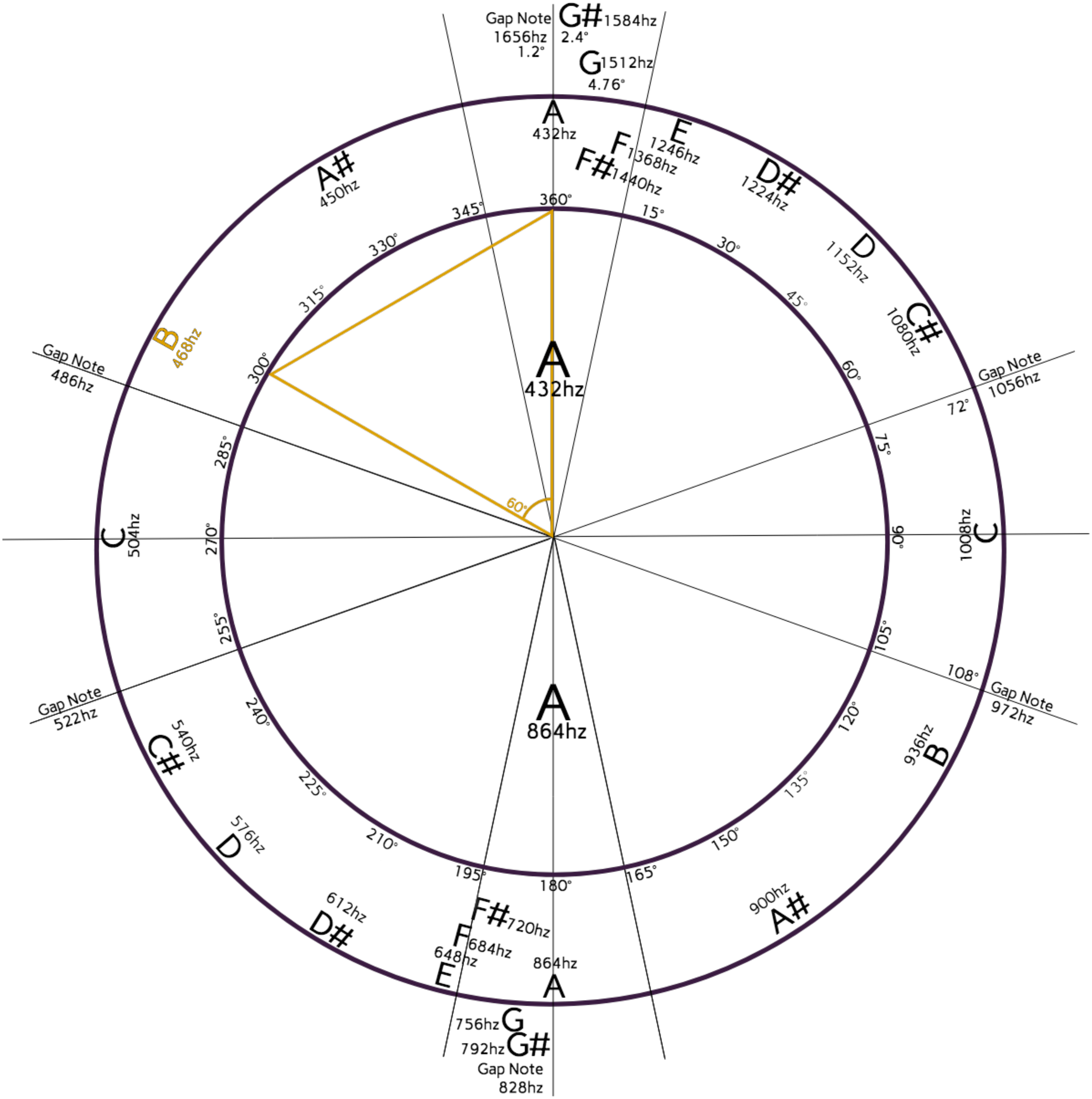
Notes F, F#, G, and G# correlate to polygons that are not depicted above
as they closely resemble circles (48, 57, 75, and 150-sided polygons)

Figure 2



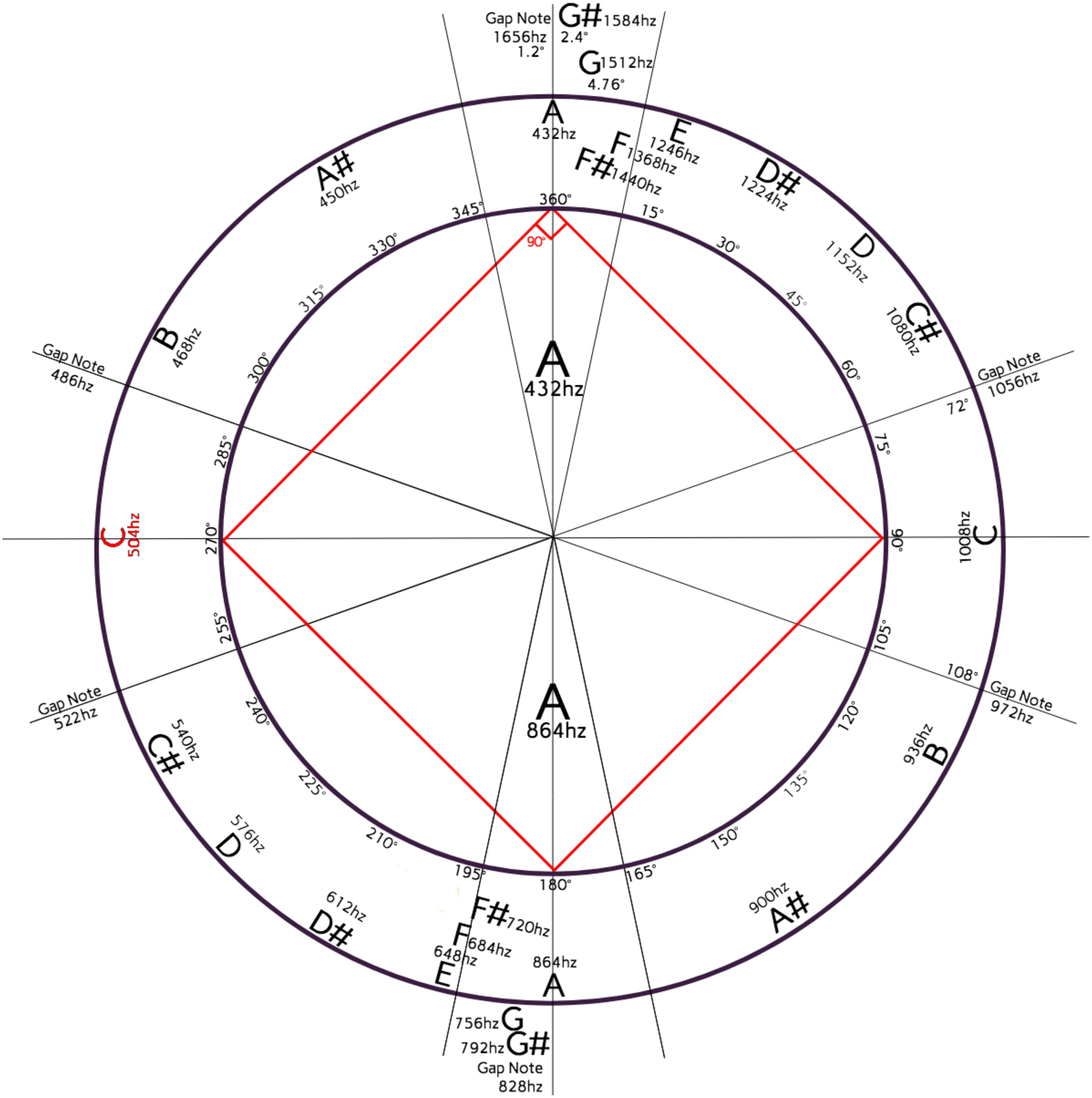
Moving a "half-step" from A – A# depicts a 30° angle, a shape commonly used as a symbol for the compass

Figure 3



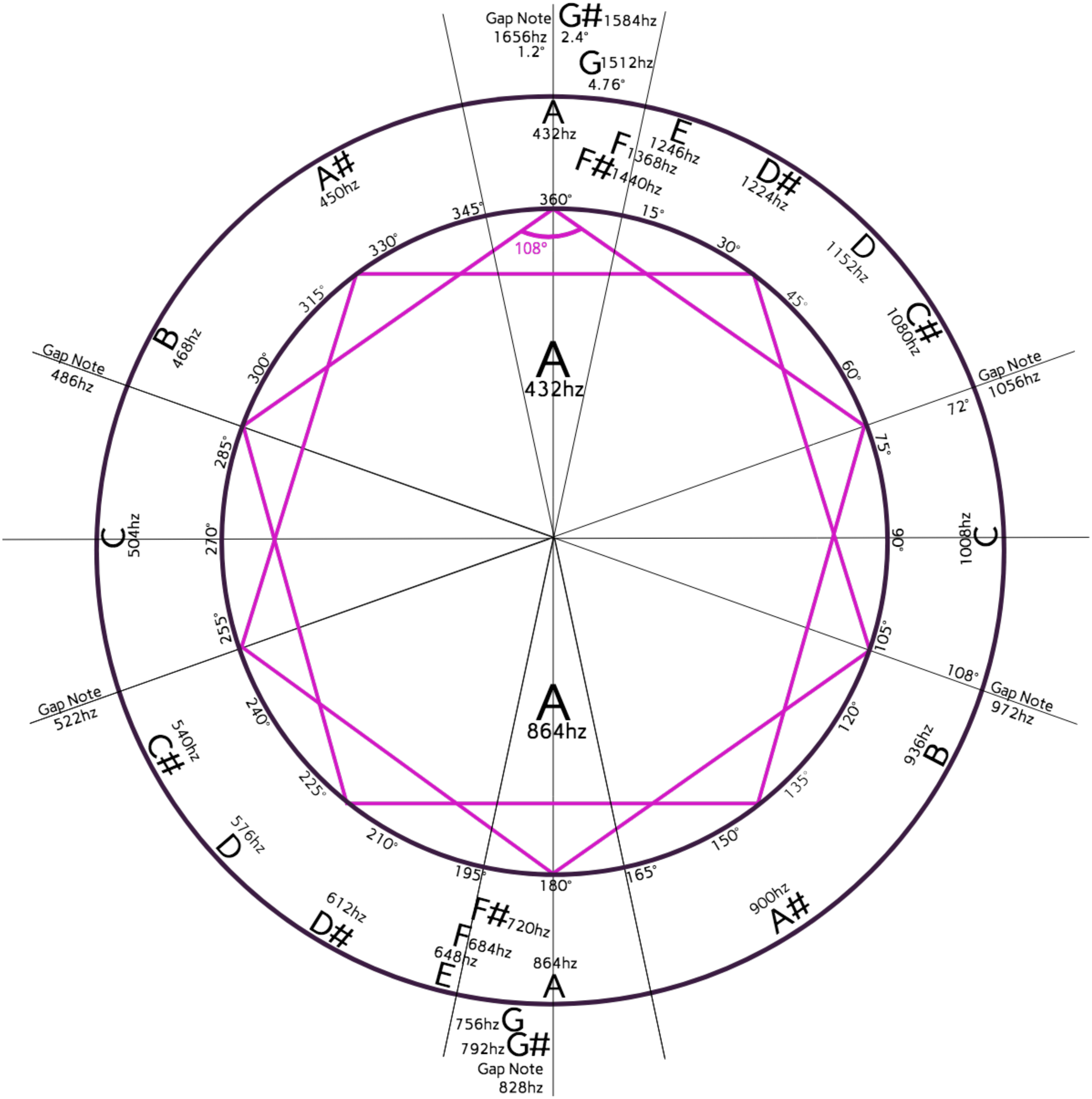
Moving a "whole-step" from A – B depicts a 60° angle, which, when closed out, forms the fundamental equilateral triangle geometry

Figure 4



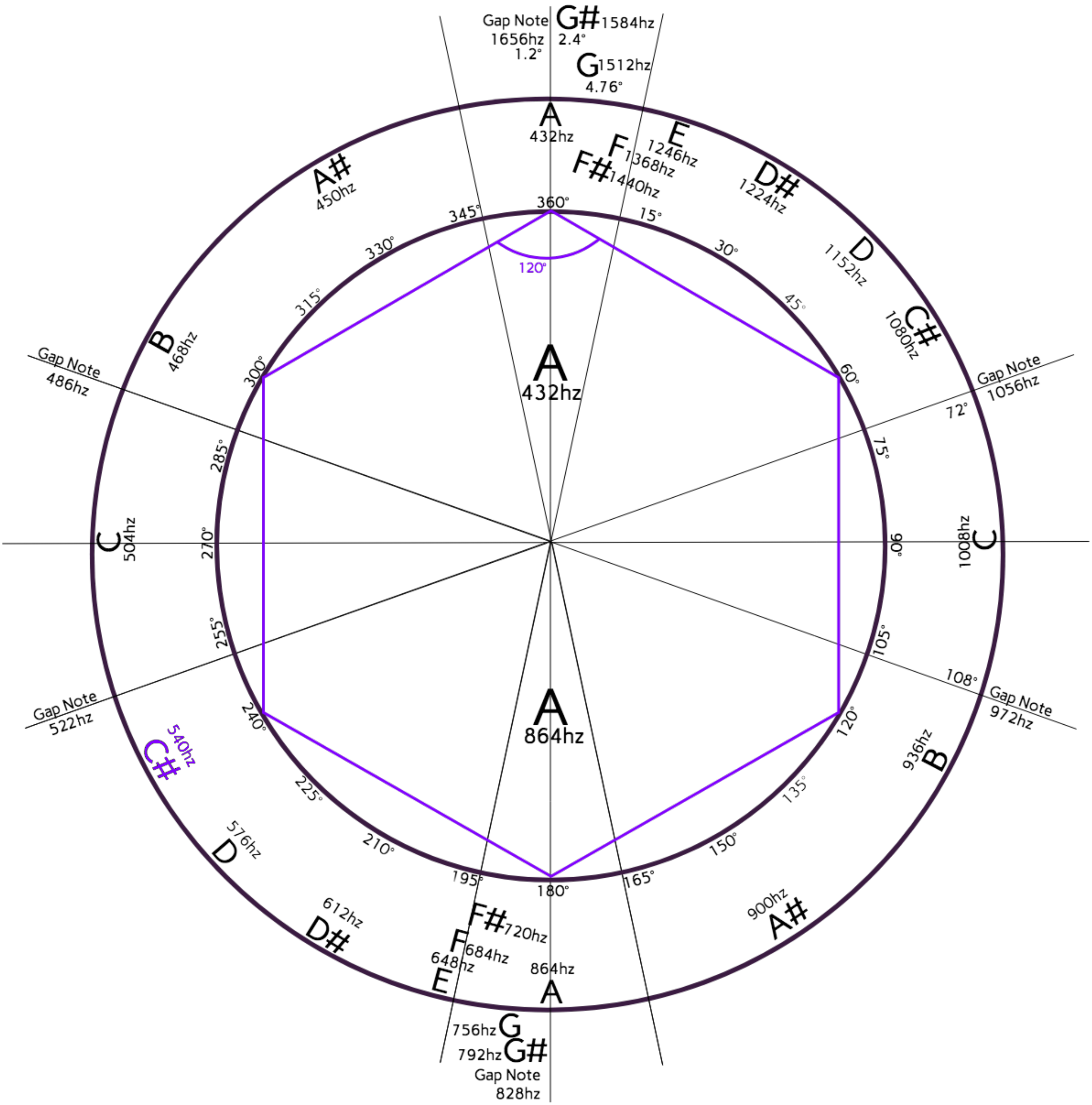
Gap Note
1656hz
1.2°
G# 1584hz
2.4°
G 1512hz
4.76°
A
432hz
F 1368hz
F# 1440hz
E 1246hz
D# 1224hz
A# 450hz
D 1152hz
C# 1080hz
B 468hz
Gap Note 486hz
Gap Note 1056hz
A
432hz
C 504hz
C 1008hz
A
864hz
Gap Note 522hz
Gap Note 972hz
C# 540hz
B 936hz
D 576hz
D# 612hz
A# 900hz
F# 720hz
F 684hz
648hz E
864hz
A
756hz G
792hz G#
Gap Note
828hz
360°
15°
30°
45°
60°
72°
75°
90°
105°
108°
120°
135°
150°
165°
180°
195°
210°
225°
240°
255°
270°
285°
300°
315°
330°
345°
90°

Figure 5



Pentagons relate to "dissonant" or enharmonic notes (like B#), which are not used in the classical chromatic scale of music

Figure 6



Gap Note
1656hz
1.2°
G#
1584hz
2.4°
G
1512hz
4.76°
A
432hz
F
1368hz
F#
1440hz
E
1246hz
D#
1224hz
A#
450hz
360°
345°
15°
330°
30°
120°
D
1152hz
315°
45°
C#
1080hz
B
468hz
Gap Note
486hz
300°
60°
A
432hz
Gap Note
1056hz
72°
285°
75°
C
504hz
270°
90°
C
1008hz
255°
105°
108°
Gap Note
972hz
Gap Note
522hz
240°
120°
A
864hz
540hz
C#
B
936hz
225°
135°
576hz
D
210°
150°
195°
180°
165°
F#
720hz
612hz
D#
F
684hz
648hz
E
864hz
A
900hz
A#
756hz
G
792hz
G#
Gap Note
828hz

Figure 7

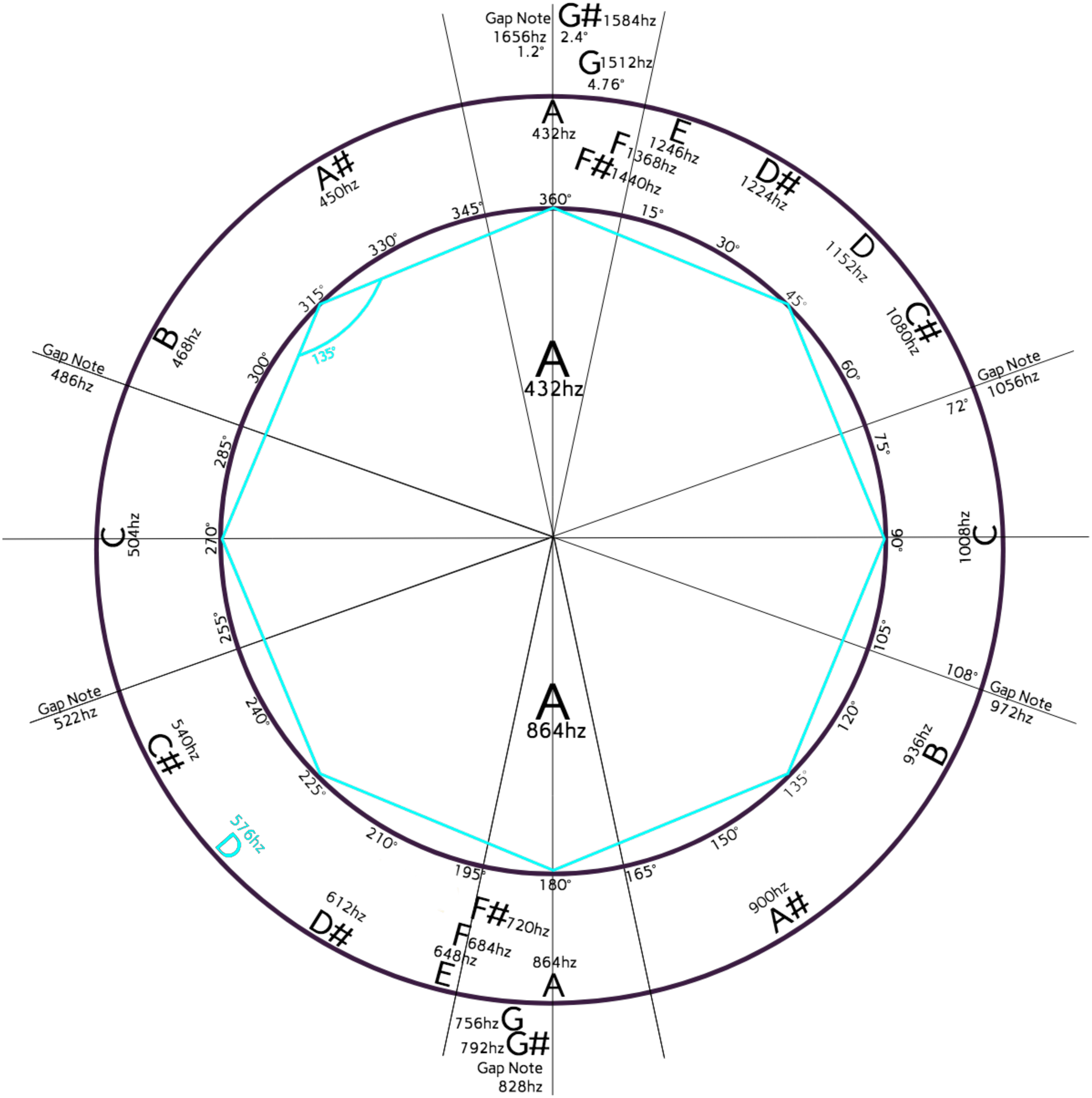

Gap Note
1656hz
1.2°
G# 1584hz
2.4°
G 1512hz
4.76°
A
432hz
F 1368hz
E 1246hz
F# 1440hz
D# 1224hz
D 1152hz
C# 1080hz
Gap Note 1056hz
72°
C 1008hz
Gap Note 972hz
108°
B 936hz
A# 900hz
A# 450hz
B 468hz
Gap Note 486hz
C 504hz
Gap Note 522hz
C# 540hz
D 576hz
D# 612hz
E 648hz
F 684hz
F# 720hz
864hz
A
756hz G
792hz G#
Gap Note
828hz
A
432hz
A
864hz
135°
360°
15°
30°
45°
60°
75°
90°
105°
120°
135°
150°
165°
180°
195°
210°
225°
240°
255°
270°
285°
300°
315°
330°
345°

Figure 8

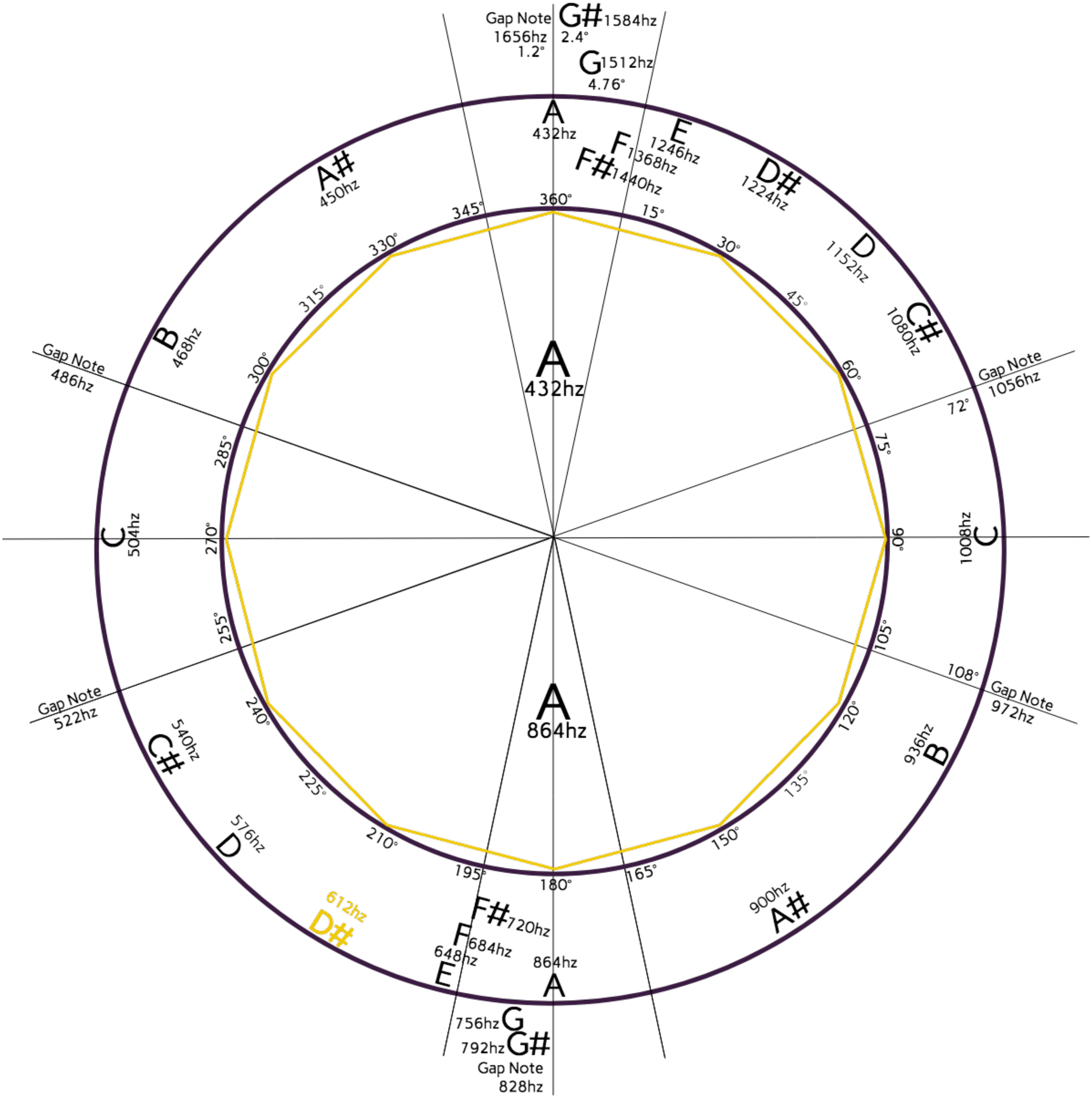
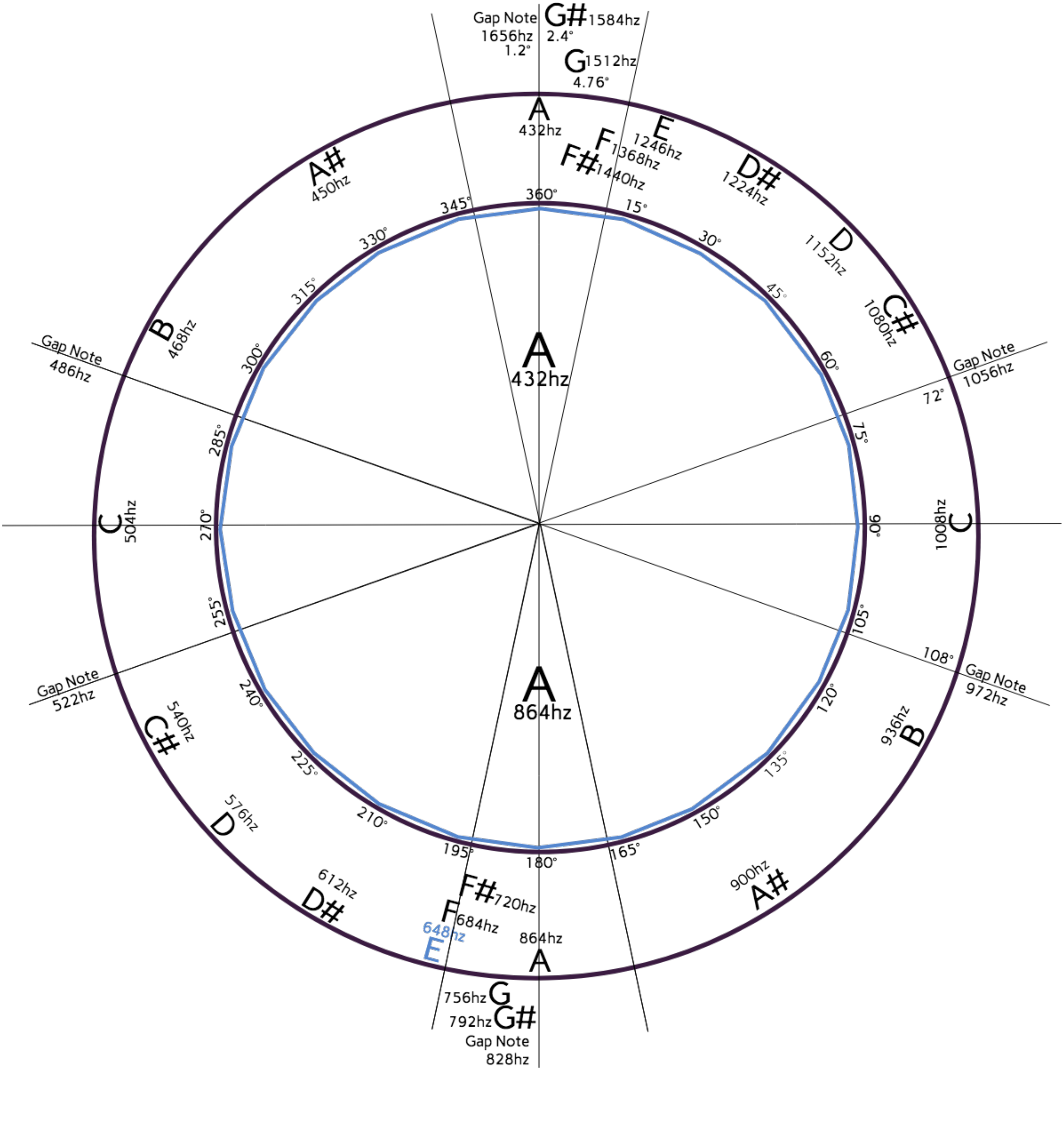

Gap Note
1656hz
1.2°
G# 1584hz
2.4°
G 1512hz
4.76°
A
432hz
F 1368hz
F# 1440hz
E
1246hz
D#
1224hz
D
1152hz
C#
1080hz
Gap Note
1056hz
72°
C
1008hz
108°
Gap Note
972hz
B
936hz
A#
900hz
A#
450hz
B
468hz
Gap Note
486hz
C
504hz
Gap Note
522hz
C#
540hz
D
576hz
612hz
D#
F# 720hz
F 684hz
648hz
E
864hz
A
756hz G
792hz G#
Gap Note
828hz
360°
15°
30°
45°
60°
75°
90°
105°
120°
135°
150°
165°
180°
195°
210°
225°
240°
255°
270°
285°
300°
315°
330°
345°
A
432hz
A
864hz

Figure 9


Gap Note
1656hz
1.2°
G# 1584hz
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A
432hz
E
1246hz
F
1368hz
F#
1440hz
D#
1224hz
D
1152hz
C#
1080hz
A#
450hz
B
468hz
Gap Note
486hz
Gap Note
1056hz
72°
C
504hz
C
1008hz
Gap Note
522hz
Gap Note
972hz
108°
C#
540hz
B
936hz
D
576hz
D#
612hz
A#
900hz
F#
720hz
F
684hz
648hz
E
864hz
A
756hz G
792hz G#
Gap Note
828hz
A
432hz
A
864hz
360°
15°
30°
45°
60°
75°
90°
105°
120°
135°
150°
165°
180°
195°
210°
225°
240°
255°
270°
285°
300°
315°
330°
345°

Figure 10